Chapter 3

Time Value of Money

“A bird in the hand is worth two in the bush”
A folklore saying

Learning Outcomes

Upon completion of this chapter, you will be able:

1. To evaluate the significance of the time value of money.
2. To understand the factors that influence the time value of money.
3. To calculate the present value of a future sum of money or a stream of future returns.
4. To explain the inverse relationship of the present value with interest rate (or discount factor) and time.
5. To demonstrate the calculation of an annuity and of a perpetuity.

Preview

We saw in Chapter 1 and 2 that investment decisions of individuals and businesses are forward-looking. In other words, the outcomes of the decisions made today (such as the decision to invest in a portfolio of securities or an investment project for a new plant or a new machinery, etc) will materialize in some future time. For example the benefits (such as returns, dividend, capital gains, revenues, profits, etc) will be received in some future time. But, these future returns are not worth the same as the money spent today. This presents a problem for the “rational” investor or “rational” firm that has to compare the additional costs of any decision/action with the additional benefits accrued from that decision/action. It becomes clear that there is a time value of money dimension that has to be taken into consideration in evaluating such investment decisions.

A Greek Cypriot folklore expression says: «Κάλλιον πέντε και στο χέρι, παρά δέκα και καρτέρει» which loosely translated to English says “Better have five in the hand than ten for which you have to wait.” This of course is equivalent to the standard English expression: “A bird in the hand is worth two in the bush”

We take up in this chapter the concept of time value of money and related concepts such as discounting, present value, future value, and annuities. We will focus mostly on understanding the rationale and methodology of discounting and calculating the present value of a future sum of money by using the compound and discount tables as presented in Appendix 1 through Appendix 4 at the end of the book. There are, of course, nowadays many computer programmes (e.g., Excel) and hand-held financial calculators that make it easy to calculate present values.
Interest Rates, Compounding and the Future Value of Money

Though we will be mostly concerned with discounting future sums and with the present value concept, the best way to understand these concepts is to first understand the critical role of interest rates in this process. Therefore, we start with the concept of compounding and the future value of sums of money. As the folklore saying above implies, people value having a sum of money today more than having the same amount in some future period. This is why when you borrow a sum of money today, the lender (say the bank) requires you to pay back more than the initial sum. Part of this is to cover the cost of service for providing the loan (including a profit), but much has to do with recovering the loss of purchasing power of the initial sum, what is the opportunity cost of money, or the time value of money.

This time value of money depends on the interest rate, the percentage of additional money you have to repay over and above the initial money borrowed (the principal). Of course, we can look at things from the reverse, from the perspective that you are not a borrower but you are a saver. In an analogous way, you expect to receive in some future time not only the sum of money initially deposited, but also an additional sum which will compensate you for not using your money now, for foregoing the use of your money for current consumption. You require, in other words, to be paid an interest rate, the percentage of additional money you require to receive over and above the initial money saved (the principal).

Example 3.1: Calculation of the Future Value of a Sum

Let’s consider an example. Assume that you put €1,000 in a bank account at an interest rate of 10% (it is wishful thinking, of course, that you can earn 10% on a savings account … but let’s amuse ourselves)! So, the initial principal amount today (or present value, PV, of the sum of money) is €1,000. How much would your account balance be in one year? Obviously the sum would be the initial principal of €1,000 plus the interest earned on this amount at 10% (i.e., €100). The balance in other words will be €1,100. Here is how we derive the answer:

\[ FV_1 = PV + PV(r) = 1000 + 1000(0.10) = 1000 + 100 = 1,100 \]

If you now decide to leave your money for a second year at the bank, and the interest rate remains at 10%, the calculations would be:

\[ FV_2 = PV + PV(r) + [PV(r)](r) = 1000 + 1000(0.10) + 1000(0.10)(0.10) \]
\[ = 1,000 + 100 + 110 = 1,210 \]

So, at the end of the second year you would have a sum of money equal to €1,210. Let’s take this process to a third stage and assume that you leave your money untouched for a third year with the same interest rate. At the end of the third year, your account balance will be:

\[ FV_3 = PV + PV(r) + [PV(r)](r) + [(PV(r))(r)](r) \]
\[ = 1000 + 1000(0.10) + 1000(0.10)(0.10) + 1000(0.10)(0.10)(0.10) \]
\[ = 1,000 + 100 + 110 + 121 \]
\[ = €1,331 \]

We can generalize the future value for a sum of money (or deposit in our example) to be received in \( t \) periods in the future with the following formula:

\[ FV = PV (1 + r)^t \]

where:
- \( FV \) = the future value (the cumulative balance of the bank account)
- \( PV \) = present value (the initial sum of money deposited in the bank)
- \( t \) = the number of periods in the future over which the interest income is earned
- \( r \) = the interest rate on the deposits

Let’s verify that this last formula when applied to an initial sum of €1000 at an interest rate of 10% will indeed give us after 3 years the value of €1,331 found above:
\[ FV = 1,000(1 + 0.10)^3 = 1,000(1.10)^3 = €1,331 \]

The term in parenthesis in the FV formula, that is \((1+r)^t\), is what we call the future value interest factor (FVIF) which we can easily find in interest factor tables such as Appendix 3 for different values of \(i\) and \(t\), that is \((FVIF_{r,t})\). To demonstrate how we find the FVIF from interest factor tables, we reproduce in Table 3.1 a section of Appendix 3 that includes the relevant coordinates, namely discount or interest rate \((r = 10\%)\) and relevant time period \((t = 3)\).

<table>
<thead>
<tr>
<th>(t)</th>
<th>1%</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
<th>8%</th>
<th>9%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0100</td>
<td>1.0200</td>
<td>1.0300</td>
<td>1.0400</td>
<td>1.0500</td>
<td>1.0600</td>
<td>1.0700</td>
<td>1.0800</td>
<td>1.0900</td>
<td>1.1000</td>
</tr>
<tr>
<td>2</td>
<td>1.0201</td>
<td>1.0404</td>
<td>1.0609</td>
<td>1.0816</td>
<td>1.1025</td>
<td>1.1236</td>
<td>1.1449</td>
<td>1.1664</td>
<td>1.1881</td>
<td>1.2100</td>
</tr>
<tr>
<td>3</td>
<td>1.0303</td>
<td>1.0612</td>
<td>1.0927</td>
<td>1.1249</td>
<td>1.1576</td>
<td>1.1910</td>
<td>1.2250</td>
<td>1.2597</td>
<td>1.2950</td>
<td>1.3310</td>
</tr>
<tr>
<td>4</td>
<td>1.0406</td>
<td>1.0824</td>
<td>1.1255</td>
<td>1.1699</td>
<td>1.2155</td>
<td>1.2625</td>
<td>1.3108</td>
<td>1.3605</td>
<td>1.4116</td>
<td>1.4641</td>
</tr>
<tr>
<td>5</td>
<td>1.0510</td>
<td>1.1041</td>
<td>1.1593</td>
<td>1.2167</td>
<td>1.2763</td>
<td>1.3382</td>
<td>1.4026</td>
<td>1.4693</td>
<td>1.5386</td>
<td>1.6105</td>
</tr>
</tbody>
</table>

Notice that by going to the 10% column and moving down to find the row \(t = 3\) (or alternatively go to to row 3 and move to the right until we find the column for \(r = 10\%\)) we find that the FVIF \((r, t = 3)\) = 1.3310. Remember that this is the value representing the term in the parenthesis in the FV formula above. Therefore, we multiply this value (1.3310) times the present value of the initial deposit \(€1,000\) to get the future value:

\[ FV = €1,000 \times 1.3310 = €1,331 \]

**Future Value of a Stream of Cash Flows**

In the event that there is a series of different sums of money deposited for different times in the future at different interest rates (a realistic scenario since banks do pay different interest rates for different sums (small vs. large deposits), and for different time periods (long-term vs. short-term deposits), then we can generalize the future value formula to account for a stream of flows (or deposits in our example) for \(t\) periods in the future as follows:

\[ FV = \sum_{t=1}^{n} PV(1 + r)^t \]

where:
- \(FV_t\) = the future value (the cumulative balance of the bank account)
- \(PV\) = present value (the different sums of money deposited in the bank)
- \(t\) = is the number of years the deposits are “locked” and interest is earned
- \(r\) = the different interest rates for each type of deposits, and
- \(\Sigma\) = is the summation operator (the Greek capital letter “sigma”).

**Example: The Power of Compounding**

To understand the power of compounding, consider that you go today, say at age 25, to your investment adviser, James Dimitriou of Axia Financial Advisers, and put a sum of €10,000 in an investment account which gives you an average return of 10% per year. Then, without touching that money (the principal or the interest income) for the next 50 years (!?), at age 75 you will have accumulated a fortune of €1,173,900 (yes, over one million Euros!).

Just to impress you more, consider now that (somehow!) James Dimitriou, your financial adviser, manages to secure for you a 20% annual return. Then, in 50 years your initial €10,000 will be worth €91,004,000 (yes, over ninety-one million Euros!).
Alternatively, you may decide to put every month €100 (or €1,200 per year) in a high-yield mutual fund earning on average 10% per year. Assuming the same life span as before, when you are 75 years old you will have accumulated in your account close to €1.4 million (€1,396,680 to be exact).

➡️ Moral of the story: it is not early enough to start saving for your retirement, no matter how difficult it is for you (say at 25 years old) to get in that frame of mind.

The Rule of 72

“The Rule of 72” is a very simple and useful “rule of thumb” that is used to determine the time (number of years) it takes for an investment to double its value. To find this, we simply divide the number 72 by the relevant interest rate.

For example, if the interest rate is 6%, using the rule of 72 we find that an investment of €1,000 will double in value in 12 years: 72 / 6 = 12. In other words, in 12 years the initial €1,000 will accumulate to €2,000. Let’s verify this using the more sophisticated annual compounding formula. The future interest rate factor for 12 years at 6% is 2.012. Thus: €1,000 * 2.012 = €2,012. This is indeed very close to the €2,000 found by the rule of thumb. If the market interest rate is 10%, then using the rule of 72, we find that the investment will double in 7.2 years. This is indeed very close to the answer found using the annual compounding formula (where the interest factor for 10% and 7.2 years) is 1.990. Thus, the FV = €1,000 * 1.990 = €1,990.

Discounting and the Present Value of a Sum

The present value is the inverse of the future value. We look at things in reverse. We start with sums of money received in some future period and seek to calculate what these sums of money are worth today. For example, how much do you need to put in the bank today at 10% interest if you want to have in one year a sum of €1,100? The obvious answer that follows from our previous calculations is €1,000. We can ask effectively the same question in a slightly different way: How much is €1,100 that you will have next year worth today if the going interest rate is 10%? The answer of course is again €1,000.

The formula for the present value is derived directly from the future value formula by solving for PV. Recall that the future value formula is \( FV = PV (1+r)^t \). By solving this equation for PV we get:

\[
P.V. = \frac{F.V}{(1+r)^t} = F.V \left[ \frac{1}{(1+r)^t} \right]
\]

where

- \( PV \) = present value (the initial sum deposited in the bank)
- \( F.V \) = future value (the cumulative balance of the bank account)
- \( t \) = the number of periods in the future over which interest income is earned
- \( r \) = the interest rate on the deposit

Example 3.2: Calculation of the Present Value

In terms of the bank account example, if you want to accumulate in your account a sum of €1,331 over three years, without withdrawing any money, at an interest rate of 10%, obviously its present value (what it is worth today, or what you need to initially deposit in your account) is much less. Applying the PV formula, we’ll find this sum to be just €1,000.

\[
P.V. = 1,331 \left[ \frac{1}{(1+0.10)^3} \right] = 1,000
\]

Again, we can think of the €1,000 as the present sum of money (the principal) that you need to put in a bank earning 10% so that you accumulate in three years a sum of €1,331. This is the future value thinking.
The term in square brackets in the PV formula (that is, \(1/(1+r)^t\)) is the present value interest factor (PVIF), which we can easily find in interest factor statistical tables such as Appendix 1 for different values of \(i\) and \(t\), that is (PVIF\(_{r,t}\)). To demonstrate how we find the PVIF from interest factor tables, we reproduce in Table 3.2 a section of Appendix 1 that includes the relevant coordinates, namely discount or interest rate which is 10% \((r = 10\%)\) and relevant time, the number of years in the future that the sum of money will be received, \((t = 3)\).

### Table 3.2: Present value of €1 \((t =1\) to \(5\) and \(r=1\) to \(10\))

<table>
<thead>
<tr>
<th>(t)</th>
<th>1%</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
<th>8%</th>
<th>9%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.990</td>
<td>0.980</td>
<td>0.971</td>
<td>0.962</td>
<td>0.952</td>
<td>0.943</td>
<td>0.935</td>
<td>0.926</td>
<td>0.917</td>
<td>0.909</td>
</tr>
<tr>
<td>2</td>
<td>0.980</td>
<td>0.961</td>
<td>0.943</td>
<td>0.925</td>
<td>0.907</td>
<td>0.890</td>
<td>0.873</td>
<td>0.857</td>
<td>0.842</td>
<td>0.826</td>
</tr>
<tr>
<td>3</td>
<td>0.972</td>
<td>0.942</td>
<td>0.915</td>
<td>0.889</td>
<td>0.864</td>
<td>0.840</td>
<td>0.816</td>
<td>0.794</td>
<td>0.772</td>
<td>0.7513</td>
</tr>
<tr>
<td>4</td>
<td>0.961</td>
<td>0.924</td>
<td>0.888</td>
<td>0.855</td>
<td>0.823</td>
<td>0.792</td>
<td>0.763</td>
<td>0.735</td>
<td>0.708</td>
<td>0.683</td>
</tr>
<tr>
<td>5</td>
<td>0.952</td>
<td>0.906</td>
<td>0.863</td>
<td>0.822</td>
<td>0.784</td>
<td>0.747</td>
<td>0.713</td>
<td>0.681</td>
<td>0.650</td>
<td>0.621</td>
</tr>
</tbody>
</table>

Notice that by going to the 10% column and moving down to find the row \(t = 3\) we find that the PVIF\(_{10,3}\) = 0.7513. Remember that this is the value representing the term in the square brackets in the PV formula above, namely \(1/(1+r)^t\). Therefore, we multiply this value (0.7513) times the future value to be received in three years (€1,331) to get the present value:

\[
PV = €1,331 \times 0.7513 = €999.99 \text{ or } €1,000
\]

The generalized formula for the present value for a series of expected future cash flows or returns from an investment (whether an individual’s portfolio investment or a firm’s investment in a business project) is given by:

\[
PV = \sum_{t=1}^{n} \frac{FV_t}{(1 + r)^t}
\]

where, as before,
- \(PV\) = present value (the initial sum deposited in the bank)
- \(FV_t\) = the future value (the cumulative sum to be received in some future period)
- \(t\) = is the number of periods in the future over which the interest income is earned
- \(r\) = the interest rate on the deposit, and
- \(\Sigma\) = is the summation operator (the Greek capital letter “sigma”).

Notice that since the discount rate \(r\) (which for practical purposes is usually represented by the interest rate) and the time variable \(t\) are in the denominator of the formula, the present value of a sum of money (or the sum of a stream of money) is inversely related to the interest rate and to time. In other words, with all other things constant, say at a given \(t\), as the interest rate increases the present value falls. Also, at a given interest rate \(r\), as time increases the present value gets smaller.

These relationships between \(r\) and \(t\), on the one hand, and PV on the other hand are shown in Figure 3.1. We see, for instance that the PV of €100 to be received in 5 years at 5% interest is about €78 (since the PVIF\(_{5.5}\) = 0.784 as found in Appendix 1) and is represented by point A in Figure 3.1, which lies on the r=5% curve at the level of 5 years on the horizontal axis (the upper curve). On the other hand, point B represents the PV of €100 still at 5% but when the sum of €100 is received in 20 year from now. The PV is shown to be just below €40 (€37.7 to be exact).

Thus, holding the interest rate constant at 5% (the upper curve) and allowing time to vary we see the inverse relationship between PV and time \((t)\). In other words:
when \( t=5 \) \( \Rightarrow \) \( PV = €78.4 \),
when \( t=10 \) \( \Rightarrow \) \( PV = €61.4 \), and
when \( t=20 \) \( \Rightarrow \) \( PV = €37.7 \)

**Figure 3.1: Present Value of a Sum of €100 for Different \( r \) and \( t \)**

If we now hold time constant, say at \( t=5 \) (\( t \) is on the horizontal axis), when \( r=10 \) (represented by the middle curve) the PV is about €62 (PVIF\(_{5,10} = 0.621\)), whereas when \( r=20 \) (the lower curve) the PV is about €40 (PVIF\(_{5,20} = 0.402\)).

You can verify this by looking at the section of Appendix 1 as shown in Table 3.2. The present value interest factor (PVIF) becomes smaller for a given time period as the interest rate increases. For example, by looking at row \( t=3 \) in Table 3.2, we see that the PVIF of €100 will start at €97.2 for 1% interest rate (point A) and it becomes smaller as we move to the right, in other words for higher and higher interest rates. For example, at 8% the PV of €100 is about €80 (€79.4 to be exact) as shown by point B, while at interest rate of 20% the PV of €100 (at \( t=3 \)) is €57.9 (point C). We show this inverse relationship between PV of €100 and interest rate in Figure 3.2 (with \( t=3 \)), while in Figure 3.3 we present this inverse relationship for different time periods (\( t = 1, 5, 10, 20 \)).

**Figure 3.2: Present Value of €100 with \( t = 3 \) and \( r \) from 1% to 20%**
With regard to the timing of receiving the future returns, as determined by the value of \( t \), it can also be verified from the section of Appendix 1 shown in Table 3.2 that at a given interest rate as time increases the PVIF gets smaller. For example, looking at \( r = 10 \) (last column), the PVIF is 0.909 for \( t = 1 \), and it becomes smaller and smaller as \( t \) increases. For example, for \( t = 5 \), the PVIF becomes 0.621. Intuitively, this means that the longer into the future (greater \( t \)) a sum of money is to be received, the smaller will be its present worth (value).

**Case Study 3.1: Total Returns of a Stock Portfolio**

An individual wants to invest €10,000 in a portfolio of stocks listed on the Cyprus Stock Exchange. He expects that there would be a stream of dividend flows from the first year to the fifth year as follows: €500, €600, €750, €1000, and €750. The investor believes that he would be able to sell the portfolio at the end of the fifth year for €11,000. Assume an interest rate of 10%.

**Questions:** Is that a good investment? Should the investor go ahead? At first sight one may hastily decide that since there is some income and at the end the investor recovers his initial investment plus there is a “profit” of €1,000, he may be inclined to go ahead with the investment? Should he?

To answer such questions one would need to find out the PV of the future dividend flows and the cashing-out sum of the investment of €11,000? The calculations of the PV of each dividend stream over the five years and the sum collected from selling the portfolio are presented in the last column of Table 3.3. Notice that the numbers of the third column are the present value interest factors (PVIF) found in Appendix 1. So, the PV of the investment (€9,492.45) is less than the sum of money that the investor would be putting down (€10,000). The investor should not invest in this portfolio, which anyhow carries a lot of risk as far as the realization of those expected returns. He is better off investing in the risk-free (or at least lower-risk) money market earning 10%.

**Table 3.3: Present Value of a Firm’s Profit Stream**

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual Dividends</th>
<th>PVIF</th>
<th>PV of each Dividends</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>€500</td>
<td>0.9091</td>
<td>€454.55</td>
</tr>
<tr>
<td>2</td>
<td>600</td>
<td>0.8264</td>
<td>495.84</td>
</tr>
<tr>
<td>3</td>
<td>750</td>
<td>0.7513</td>
<td>563.48</td>
</tr>
<tr>
<td>4</td>
<td>1,000</td>
<td>0.6830</td>
<td>683.00</td>
</tr>
<tr>
<td>5</td>
<td>750</td>
<td>0.6209</td>
<td>465.68</td>
</tr>
<tr>
<td>Cash out</td>
<td>11,000</td>
<td>0.6209</td>
<td>6829.90</td>
</tr>
</tbody>
</table>

**Present Value of Investment = \( \sum_{i} PV_i = \) €9,492.45**

**Case Study 3.2: Value of a Firm**

In preparing the strategic plan of Omega Bank, the director of Planning and Development and the Financial Controller of the bank expect that the stream of profits over the next 5 years will be as shown in...
Table 3.4: They also expect that at the end of the sixth year the assets and “good will” of the bank would be worth €2 billion. Assume that the interest rate is 6% and it is expected to remain at that level for the whole period.

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual Profit Stream (in €million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>€100</td>
</tr>
<tr>
<td>2</td>
<td>125</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>175</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>6</td>
<td>2,000</td>
</tr>
</tbody>
</table>

Questions
1. Accepting that the above profit and interest rate scenario is true and logical, if InterBank is interested to buy Omega Bank how much would it pay today?
2. InterBank believes that the true cost of capital will be 10% over the next 6 years. How much would it be willing to offer to buy Omega Bank?

Answer to Question 1:
To answer this question, we need to calculate the sum of the present values of each stream of profits over the five-year period as well as the present value of the value of assets and good will expected in the sixth year. These calculations of the present values are shown in the last column of Table 3.5. Notice that the numbers of the third column are the present value interest factors for 6% (PVIF<sub>6</sub>), which are found in Appendix 1 in the back of the Book.

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual Profit Stream (in €million)</th>
<th>PVIF (r = 6%)</th>
<th>PV of each Profit Stream (€million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>€100</td>
<td>0.943</td>
<td>94.34</td>
</tr>
<tr>
<td>2</td>
<td>125</td>
<td>0.890</td>
<td>111.25</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>0.839</td>
<td>125.85</td>
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<td>4</td>
<td>175</td>
<td>0.792</td>
<td>138.60</td>
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<tr>
<td>5</td>
<td>200</td>
<td>0.747</td>
<td>149.40</td>
</tr>
<tr>
<td>6</td>
<td>2,000</td>
<td>0.705</td>
<td>1410.00</td>
</tr>
</tbody>
</table>

Adding the values in the last column would give us the present value of the firm as:

\[
\text{Value of Omega Bank} = \sum PV_i = €2029.44 \text{ million (or about €2 billion)}
\]

Thus, accepting that the expected profits and the interest rate4s are true and logical, the Interbank would have to pay €2 billion to buy Omega Bank.

Answer to Question 2:
To answer this question, essentially we repeat the process followed in Question 1, but now the PVIFs will be for 10%. For practice, the answer is left to you. (Hint: since the discount rate is higher, intuitively you should find that the PV is smaller than before. Recall that the PV is inversely related to the discount rate!

**Example 3.3: Excel Spreadsheet Practice for Finding the Present Value**

**Task:** Find the present value of €10,000 to be received at the end of 5 years at 10% per year.

**Procedure:** In the Formula function **fx**, enter “=PV(” and you will get the following prompt from Excel (notice the negative sign in front of PV and the opening parenthesis after PV):

\[
= PV(rate;nper;pmt;fv;type)
\]

where: rate (r) is the interest rate per period (with the % sign)

nper (f) is the number of year in the future when payment will be received
Present Value of an Annuity

Let’s first define an annuity. An *annuity* is a fixed sum of money received every year for a specified number of years. For example, the interest payments you receive on a bond (the coupon) is an annuity (to be discussed in Chapter 6). If you win a scholarship of €20,000 per year for four years, then this is an annuity as well. If you are a beneficiary of a trust entitling you to receive a series of equal annual payments of €5,000 for the following 5 years, then this is also an annuity. For sure, due to the opportunity cost of money (the time value of money) the present value of those five equal payments of €5,000 is certainly less than €25,000 (the nominal sum of €5,000 received for 5 years). Of course, you know additionally that the present value would depend on the interest rate, indeed the opportunity cost of money. But, what is the actual present value of the €5,000 annuity from the trust? There are two ways to find the solution.

**Indirect Method:** The first way is simply to use the standard PV formula for a series of payments, in which case each FV is €5,000 which is then discounted by the appropriate PVIF for the specific $t$ and $r$. We know that $t = 5$ (years). Let’s assume that the interest rate is 5% ($r = 0.05$). So from the PV interest factor tables of Appendix 1, reproduced in Table 3.6 for your convenience, we simply go to the 5% column and then go down that column for years 1 through 5.
Table 3.6: Present Value of €1 (t = 1 to 5 and r = 1 to 10). PVIF(t, r) = \[
\frac{1}{(1 + r)^t}
\]

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<thead>
<tr>
<th></th>
<th>1%</th>
<th>2%</th>
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<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
<th>8%</th>
<th>9%</th>
<th>10%</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.990</td>
<td>0.980</td>
<td>0.971</td>
<td>0.962</td>
<td>0.952</td>
<td>0.943</td>
<td>0.935</td>
<td>0.926</td>
<td>0.917</td>
<td>0.909</td>
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<td>2</td>
<td>0.980</td>
<td>0.961</td>
<td>0.943</td>
<td>0.925</td>
<td>0.907</td>
<td>0.890</td>
<td>0.873</td>
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<td>3</td>
<td>0.972</td>
<td>0.942</td>
<td>0.915</td>
<td>0.889</td>
<td>0.863</td>
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<td>0.816</td>
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<td>0.7513</td>
</tr>
<tr>
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<td>0.961</td>
<td>0.924</td>
<td>0.888</td>
<td>0.855</td>
<td>0.827</td>
<td>0.792</td>
<td>0.763</td>
<td>0.735</td>
<td>0.708</td>
<td>0.683</td>
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<tr>
<td>5</td>
<td>0.952</td>
<td>0.906</td>
<td>0.863</td>
<td>0.822</td>
<td>0.7835</td>
<td>0.747</td>
<td>0.713</td>
<td>0.681</td>
<td>0.650</td>
<td>0.621</td>
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</tbody>
</table>

So, using the indirect method, the present value of the annuity is found as follows:

\[
PV = \sum_{t=1}^{n} \frac{FV_t}{(1 + r)^t} = (5000 \times 0.9524) + (5000 \times 0.9070) + (5000 \times 0.8638) + (5000 \times 0.8227) + (5000 \times 0.7835)
\]

\[
= 4762 + 4535 + 4319 + 4113.5 + 3917.5 = €21,647
\]

You immediately recognize, of course, that the numbers in parentheses (in other words, 0.9524, 0.9070, …) that the €5,000 is multiplied with are the PVIF for a 5% discount rate.

**Direct Method:** The second way is more direct and less time-consuming than the method just shown. We use a specially constructed present value table for annuities that basically adds the PVIF sequentially for each additional year, since the sum of money received is equal for all years. Essentially the formula for the PV of an annuity is expressed as:

\[
PVA = \frac{FV}{1}\sum_{t=1}^{n} \frac{1}{(1 + r)^t}
\]

The summation of the PVIFs is what we call the present value interest factor for an annuity (PVIFA). The task of finding the present value of an annuity then is simplified significantly since all we have to do is to find the PVIFA from the appropriate tables and then multiply that by the annuity, the sum of money received each year (€5,000 in our example). We reproduce in Table 3.7 a section of Appendix 2 for the present value interest factors for an annuity, PVIFA(t, r).

Table 3.7: Present Value of €1 Annuity (t = 1 - 5 and r = 1 - 10). PVIFA(t, r) = \[
\frac{1-(1+r)^{-t}}{r}
\]

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<thead>
<tr>
<th></th>
<th>1%</th>
<th>2%</th>
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<th>4%</th>
<th>5%</th>
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<th>8%</th>
<th>9%</th>
<th>10%</th>
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<td>0.943</td>
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<td>0.926</td>
<td>0.917</td>
<td>0.909</td>
</tr>
<tr>
<td>2</td>
<td>1.970</td>
<td>1.942</td>
<td>1.913</td>
<td>1.886</td>
<td>1.859</td>
<td>1.833</td>
<td>1.808</td>
<td>1.783</td>
<td>1.759</td>
<td>1.736</td>
</tr>
<tr>
<td>3</td>
<td>2.941</td>
<td>2.884</td>
<td>2.829</td>
<td>2.775</td>
<td>2.723</td>
<td>2.673</td>
<td>2.624</td>
<td>2.577</td>
<td>2.531</td>
<td>2.487</td>
</tr>
</tbody>
</table>

Let us use the PV formula for an annuity and validate that we will indeed get the same answer as in the indirect way. We find from the above table that the present value interest factor for an annuity for 5 years at 5% (PVIFA_{5,5}) is 4.3295. So,

\[
PVA = \frac{FV}{1}\sum_{t=1}^{n} \frac{1}{(1 + r)^t} = 5000 \times 4.3295 = €21,647 \text{ (same as before) .}
\]
**Case Study 3.3: Buying vs. Leasing**

Many car dealers offer you the choice to buy or to lease cars. Assume that you are looking into the two options. The local dealer of BMW sells the new model for €20,000 or leases it for €2,500 per year for 5 years and at the end of the five-year period you can buy the car for €15,000. Assume that the interest rate on car loans is 10% and it is expected to remain at that level for the next five years.

**Question:** Is buying or leasing the best option for you?

To answer this question, we need to calculate the sum of the PV of the five installments and the PV of the selling price at the end of the five-year period. Since we have five equal installments, essentially we are faced with an annuity. From Appendix 2 we find that the PVIFA for $t = 5$ and $r = 10\%$ is 3.7908, while the PVIF in $t = 5$ and $r = 10\%$ is 0.6209. So we have the following calculations:

\[
\text{PV of leasing installments} = 2,500 \times 3.7908 = 9,477 \\
\text{PV of selling price} = 15,000 \times 0.6209 = 9,314
\]

Adding these present values we find that the PV of the leasing option is €18,791. Comparing this with the purchase price of €20,000 we see that leasing is the preferred (less expensive) option.

**Case Study 3.4: Winning the Lotto**

Assume you win €1 million in the Lotto. For simplicity let’s also assume that there is no taxation so you get the full amount. The usual practice of the Lottery Authorities is to appropriate the winnings over a number of years by advancing the winner an equal annual amount. Let’s say that the Lotto Authorities in Cyprus pay the winners in ten equal installments. In your case, since you won €1 million, you will be receiving €100,000 every year for the following 10 years. When you go to the Lotto Authorities to present your winning ticket, the authorities present you with a proposal: Accept their standard policy of paying the winnings with equal installments over 10 years, or pay you a lump sum now of €700,000. By giving you this option, the Lotto Authorities, of course, know very well that money in the future is worth less than money today.

**Questions:**
1. Which of the two propositions is more valuable at 5%? At 10%?
2. If you were to make a counter-offer which lump-sum would you recommend? Why this amount?
3. Is there a lump-sum that would basically make you indifferent as to which offer to accept?

**Present Value of Perpetuity**

A perpetuity is a special case of an annuity when the equal sum of money is received forever, in other words in perpetuity. Though it may be difficult to imagine situations where a sum of money is received indefinitely, there are situations of special types of bonds (perpetual bonds, known in the UK as consuls, which do not repay the principal at any time), or special preferred stock which are issued without a maturity (called perpetual preferred stock), or trusts that do pay a fixed amount of money indefinitely to beneficiaries and their heirs. In those situations the present value of these perpetual flows can be calculated using a simple formula:

\[
PV_{\text{Perpetuity}} = \frac{FV}{r}
\]

\[
PV_{\text{Perpetuity}} = \frac{FV}{(1 + r)^n} + \frac{FV}{(1 + r)^n} + \frac{FV}{(1 + r)^n} + \ldots + \frac{FV}{(1 + r)^n}
\]
Example 3.4: Present Value of a Perpetuity

As an example, consider that you are the privileged beneficiary of a trust left to you by a rich (very rich!) uncle that entitles you to receive €100,000 annually forever, in other words for the rest of your life and the life of your heirs! What is the value today of that newfound fortune? Well, now you have a method to calculate it by simply applying the above simple formula for the present value of perpetuity:

$$ PV_{\text{perpetuity}} = \frac{FV}{r} = \frac{100000}{0.05} = 2,000,000 $$

Not bad to discover one morning that you are a millionaire (over your life time)!

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Study Questions

Multiple Choice Questions

1. If you put €5,000 in a bank account at 5% interest, how much will you have in this account in three years?
   A. €5,5125
   B. €5,7880
   C. €6,1080
   D. €5,5000

2. If you put €500 every year in a bank account at 4% interest, approximately how much will you have in this account in 10 years?
   A. €5,000
   B. €6,000
   C. €7,000
   D. €10,000

3. The term $(1 + i)^n$ is called
   A. PVIF
   B. FVIF
   C. PVIFA
   D. FVIFA

4. With continuous compounding at 10 percent for 30 years, the future value of an initial investment of €2,000 is approximately:
   A. €34,898
   B. €40,171
   C. €164,521
   D. €328,282
5. You open a savings account that pays 4.5% annually. How much must you deposit each year in order to have €50,000 in five years?
A. €8,321
B. €9,629
C. €8,636
D. €9,140
E. €6,569

6. An annuity pays €4,000 a year for the next 20 years. The interest rates is 8% over this period. The present value of the annuity is approximately:
A. €32,562
B. €40,322
C. €39,272
D. €80,882

7. The present value of a future sum of money is lower
A. as interest rates increase and as the time of the payment increases.
B. as interest rates increase and as the time of the payment decreases.
C. as interest rates decrease and as the time of the payment increases.
D. as interest rates decrease and as the time of the payment decreases.

8. You expect to receive in three years an amount of €5,000. If the interest rate in the meantime increases, the present value of that future amount to you would
A. rise
B. fall
C. not change
D. (a) and (b) are possible

9. What is the value of a perpetuity of €400 a year if the required return is 10%?
A. €3,500
B. €4,000
C. €14,500
D. €3,200
E. €40,000

10. Andreas, George and Christina each receive €5,000 from their grandfather. Andreas puts the €5,000 in a bank for 20 years at 4% interest, George puts his €5,000 in a mutual fund for 15 years at 6% return, while Christina invests her €5,000 in the stock market for 10 years earning 10% total return. Which one will accumulate the most money in their investment account at the end of their chosen investment period?
A. Andreas
B. George
C. Christina
D. All three accumulate the same amount.

11. What is the PV of €8,000 to be paid at the end of three years if the interest rate is 11%?
A. €5,850
B. €4,872
C. €6,725
D. €1,842
E. €1,500

12. Assume that the interest rate is 5%. Which of the following cash-flows will you prefer?

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>€400</td>
<td>€300</td>
<td>€200</td>
<td>€100</td>
</tr>
</tbody>
</table>
A. €400  €300  €200  €100
B. €100 €200 €300 €400
C. €250 €250 €250 €250
D. Any of the above, since they each sum to €1,000.

13. The "Rule of 72" says that if you earn 8% per year, your money will double in:
A. 12 years
B. 6 years
C. 8 years
D. 9 years
E. 72 years

18. If the interest rate is 5 percent, which of the following has the greatest present value?
A. €1100 paid in two years
B. €500 paid in one year plus €500 paid in two years
C. €300 paid today plus €400 paid in one year plus €400 paid in two years
D. €1000 today

15. As interest rates go up, the present value of a stream of fixed cash flows
A. goes down
B. goes up
C. stays the same
D. cannot tell

16. Which of the following is the correct expression for finding the present value of a €1000 payment 5 years from today if the interest rate is 4 percent?
A. $1000 / (1.04)^5$
B. $1000 + 1000 (1.04)^5$
C. $1000 / (1.04)^5$
D. $1000 + 1000 / (1.05)^4$
E. None of the above is correct

17. You win €200,000 on the Lottery. You can receive the entire amount now or in ten equal payments of €25,000 per year starting one year from today. What should you do if the interest is 5% and is expected to remain stable over the period?
A. Take the option with installments of €25,000.
B. Take the €200,000 up front.
C. Take the option with installments of €25,000 as long as you expect the interest rate to increase over 5% in the following years
D. Cannot be determined with the information given.

**Essays, Problems and Applications**

1. What is the PVIF for i = 8% and n = 10?

2. What is the PVIFA (annuity) for i = 8% and n = 10?

3. Suppose you make an investment of €10,000. The first year the investment returns 15%, the second year it returns 2%, and the third year in returns 10%. How much would this investment be worth at the end of three years, assuming no withdrawals are made?

4. You have just won €5,000 playing the lottery. You are going to save this for your retirement in 30 years. If your investment yields 12% (and all of it is reinvested in your retirement account), how much will you have accumulated (saved) for your retirement.
5. Lionel Messi has received two offers, one from Barcelona and one from Manchester United for playing soccer. He wants to select the best offer, based on considerations of money alone. The offer from Barca is for €10m to be received as follows: €2m a year for 5 years. The offer from MU is for €12m to be received as follows: €1.5m a year for four years and €6m to be received in year 5. What would you advise Lionel Messi to do? Assume three interest rate scenarios: 5%, 8% and 10%.

6. You are a financial planner and one of your clients is about to retire. He has the option from his pension scheme to receive upon retirement a lump sum of €200,000 or an annuity of €25,000 for ten years. What would you advise your client to do (in other words, which is worth more in present value terms), if the interest rate is 5% for the whole period? Assume that no taxes are paid in either case.

7. Assume that you win €1,000,000 in the lottery and you have the choice to get €500,000 today or accept to receive €50,000 per year for the next 20 years. You believe that the interest rate will remain fairly constant at 5%. Which is the better choice for you?

8. You invest your money in a bank account at a nominal annual rate of 7.2%, compounded annually. How many years will it take for you to double your money?

9. Your long-forgotten uncle from the United States, Uncle Tom, died and left you €100,000 (!!). But there is a catch! The condition in his will states that for the next 10 years you cannot spend the money to buy goods (your favourite sports car, for example), but instead to put the money in an investment account. Let’s say that the investment earns 10% over the next 10 years. How much will you have at the end of 10 years?

10. You are twenty-five years old. You decide to start putting €1,000 per year into a retirement account and will continue to do so until you are sixty-five, for a total of 40 years. Assume that you invest in fixed income securities and the account earns 5% annually. How much will you have in the account upon your retirement?

11. Your best friend Andreas (age 25) is a smoker and spends €5 a day on cigarettes. You convince him (somehow!?) to quit this bad habit and put that money (which amounts to €5*365 days = €1825 per year) instead in an investment account earning him 10% per year. How much will he have in the account at age 65? Do you think he will be thankful for convincing him to quit smoking?

12. You deposit €10,000 in a bank account that yields 6 percent compound interest.
   a) How much will you accumulate in your account in 5 years?
   b) Using the compound formula, in how many years will your investment double?
   c) Using the “Rule of 72”, in how many years will your investment double?

13. In 2000 the average tuition for one year at European University was €2,000. Ten years later, in 2010, the average cost is about €10,000. What is the growth rate in tuition over the 10-year period? Note: Figures are hypothetical!

14. Sunrise Resorts Ltd is thinking to build a new luxury hotel complex in Limassol, Cyprus complete with marina, spa, and other modern facilities and is estimated to cost €100 million today. The Company’s financial manager estimates that the firm will have an income of €20 million per year for the following 10 years. If the expected interest over this period will remain constant at 5%, should Sunmiles Resorts undertake to build the hotel complex?

15. Venus Airways is considering putting three different routes on its scheduled flights each of which will cost it €10 million for additional airplanes and crew. Route A will generate €12 million in revenue at the end of one year. Route B will generate €15 million in revenue at the end of two years. Route C will generate €18 million in revenue at the end of three years. Which option should the firm choose?